Series

If $X := (x_n)$ is a sequence in $\mathbb{R}$, then the **infinite series** (or simply the **series**) **generated by** X is the sequence $S := (s_k)$ defined by

$$\begin{aligned} s_1 &:= x_1 \\ s_2 &:= s_1 + x_2 \quad (= x_1 + x_2) \\ &\cdots \\ s_k &:= s_{k-1} + x_k \quad (= x_1 + x_2 + \cdots + x_k) \\ &\cdots \end{aligned}$$

The numbers x_n are called the **terms** of the series and the numbers s_k are called the **partial sums** of this series. If $\lim S$ exists, we say that this series is **convergent** and call this limit the **sum** or the **value** of this series. If this limit does not exist, we say that the series S is **divergent.**

It is convenient to use symbols such as

$$\sum(x_n) \quad \text{or} \quad \sum x_n \quad \text{or} \quad \sum_{n=1}^{\infty} x_n$$

Example Consider the sequence $X := (r^n)_{n=0}^{\infty}$ where $r \in \mathbb{R}$, which generates the **geometric series**:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots . \tag{3}$$

We will show that if $|r| < 1$, then this series converges to $1/(1-r)$. (See also Example 1.2.4(f).) Indeed, if $s_n := 1 + r + r^2 + \cdots + r^n$ for $n \geq 0$, and if we multiply s_n by r and subtract the result from s_n, we obtain (after some simplification):

$$s_n(1-r) = 1 - r^{n+1}.$$

Therefore, we have

$$s_n - \frac{1}{1-r} = \frac{r^{n+1}}{1-r},$$

from which it follows that

$$\left| s_n - \frac{1}{1-r} \right| \leq \frac{|r|^{n+1}}{|1-r|}.$$

Since $|r|^{n+1} \to 0$ when $|r| < 1$, it follows that the geometric series converges to $1/(1-r)$ when $|r| < 1$.

Example Consider the series generated by $((-1)^n)_{n=0}^{\infty}$; that is, the series:

$$\sum_{n=0}^{\infty}(-1)^n = (+1)+(-1)+(+1)+(-1)+\cdots. \tag{4}$$

It is easily seen (by Mathematical Induction) that $s_n = 1$ if $n \geq 0$ is even and $s_n = 0$ if n is odd; therefore, the sequence of partial sums is $(1, 0, 1, 0, \ldots)$. Since this sequence is not convergent, the series (4) is divergent.

The *n*th Term Test *If the series* $\sum x_n$ *converges, then* $\lim(x_n) = 0$.

Proof. By Definition , the convergence of $\sum x_n$ requires that $\lim(s_k)$ exists. Since $x_n = s_n - s_{n-1}$, then $\lim(x_n) = \lim(s_n) - \lim(s_{n-1}) = 0$.

Example

Consider the series $\sum_{n=1}^{\infty}\frac{1}{n}$.

We know that the sequence $\{S_n\}$ where $S_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots\ldots\ldots+\frac{1}{n}$ is divergent

therefore $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent series, although $\lim_{n\to\infty} a_n = 0$.

This implies that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent.

It is know as basic divergent test.

Theorem (General Principle of Convergence)

A series $\sum a_n$ is convergent if and only if for any real number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left|\sum_{i=m+1}^{\infty} a_i\right| < \varepsilon \qquad \forall\ n > m > n_0$$

Proof

Let $S_n = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots + a_n$

then $\{S_n\}$ is convergent if and only if for $\varepsilon > 0$ $\exists$ a positive integer n_0 such that

$$|\ S_n - S_m\ | < \varepsilon \qquad \forall\ n > m > n_0$$

$$\Rightarrow \left|\sum_{i=m+1}^{\infty} a_i\right| = |\ S_n - S_m\ | < \varepsilon$$

Hence Proved

Theorem

Let $\sum a_n$ be an infinite series of non-negative terms and let $\{S_n\}$ be a sequence of its partial sums then $\sum a_n$ is convergent if $\{S_n\}$ is bounded and it diverges if $\{S_n\}$ is unbounded.

Proof

Since $a_n \geq 0 \quad \forall \ n \geq 0$

$$S_n = S_{n-1} + a_n > S_{n-1} \qquad \forall \ n \geq 0$$

therefore the sequence $\{S_n\}$ is monotonic increasing and hence it is converges if $\{S_n\}$ is bounded and it will diverge if it is unbounded.

Hence we conclude that $\sum a_n$ is convergent if $\{S_n\}$ is bounded and it divergent if $\{S_n\}$ is unbounded.

Theorem (Comparison Test)

Suppose $\sum a_n$ and $\sum b_n$ are infinite series such that $a_n > 0$, $b_n > 0 \ \forall \ n$. Also suppose that for a fixed positive number λ and positive integer k, $a_n < \lambda b_n \ \ \forall \ n \geq k$

Then $\sum a_n$ converges if $\sum b_n$ is converges and $\sum b_n$ is diverges if $\sum a_n$ is diverges.

Proof

Suppose $\sum b_n$ is convergent and

$$a_n < \lambda b_n \quad \forall \ n \geq k \ \ \ldots\ldots\ldots\ldots \ (i)$$

then for any positive number $\varepsilon > 0$ there exists n_0 such that

$$\sum_{i=m+1}^{n} b_i < \frac{\varepsilon}{\lambda} \qquad n > m > n_0$$

from (i)

$$\Rightarrow \sum_{i=m+1}^{n} a_i < \lambda \sum_{i=m+1}^{n} b_i < \varepsilon \quad , \quad n > m > n_0$$

$$\Rightarrow \sum a_n \text{ is convergent.}$$

Now suppose $\sum a_n$ is divergent then $\{S_n\}$ is unbounded.

$\Rightarrow \ \exists$ a real number $\beta > 0$ such that

$$\sum_{i=m+1}^{n} b_i > \lambda \beta \qquad , \quad n > m$$

from (i)

$$\Rightarrow \sum_{i=m+1}^{n} b_i > \frac{1}{\lambda} \sum_{i=m+1}^{n} a_i > \beta \quad , \quad n > m$$

$$\Rightarrow \sum b_n \text{ is convergent.}$$

Example

We know that $\sum \frac{1}{n}$ is divergent and

$$n \geq \sqrt{n} \qquad \forall \ \ n \geq 1$$

$$\Rightarrow \frac{1}{n} \leq \frac{1}{\sqrt{n}}$$

$$\Rightarrow \sum \frac{1}{\sqrt{n}} \text{ is divergent as } \sum \frac{1}{n} \text{ is divergent.}$$

Example The Series $\sum \frac{1}{n^{\alpha}}$ is convergent if $\alpha > 1$ and diverges if $\alpha \leq 1$.

Let $S_n = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + ... + \frac{1}{n^\alpha}$

If $\alpha > 1$ then

$$S_n < S_{2n} \quad \text{and} \quad \frac{1}{n^\alpha} < \frac{1}{(n-1)^\alpha}$$

Now $S_{2n} = \left[1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \frac{1}{4^\alpha} .... + \frac{1}{(2n)^\alpha}\right]$

$$= \left[1 + \frac{1}{3^\alpha} + \frac{1}{5^\alpha} + ... + \frac{1}{(2n-1)^\alpha}\right] + \left[\frac{1}{2^\alpha} + \frac{1}{4^\alpha} + \frac{1}{6^\alpha} + ... + \frac{1}{(2n)^\alpha}\right]$$

$$= \left[1 + \frac{1}{3^\alpha} + \frac{1}{5^\alpha} + ... + \frac{1}{(2n-1)^\alpha}\right] + \frac{1}{2^\alpha}\left[1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + ... + \frac{1}{(n)^\alpha}\right]$$

$$< \left[1 + \frac{1}{2^\alpha} + \frac{1}{4^\alpha} + ... + \frac{1}{(2n-2)^\alpha}\right] + \frac{1}{2^\alpha} S_n$$

replacing 3 by 2, 5 by 4 and so on.

$$= 1 + \frac{1}{2^\alpha}\left[1 + \frac{1}{2^\alpha} + ... + \frac{1}{(n-1)^\alpha}\right] + \frac{1}{2^\alpha} S_n$$

$$= 1 + \frac{1}{2^\alpha} S_{n-1} + \frac{1}{2^\alpha} S_n \quad = 1 + \frac{1}{2^\alpha} S_{2n} + \frac{1}{2^\alpha} S_{2n} \qquad \because S_{n-1} < S_n < S_{2n}$$

$$= 1 + \frac{2}{2^\alpha} S_{2n}$$

$$\Rightarrow \quad S_{2n} < 1 + \frac{1}{2^{\alpha-1}} S_{2n}$$

$$\Rightarrow \left(1 - \frac{1}{2^{\alpha-1}}\right) S_{2n} < 1 \quad \Rightarrow \left(\frac{2^{\alpha-1} - 1}{2^{\alpha-1}}\right) S_{2n} < 1 \quad \Rightarrow \quad S_{2n} < \frac{2^{\alpha-1}}{2^{\alpha-1} - 1}$$

i.e. $S_n < S_{2n} < \frac{2^{\alpha-1}}{2^{\alpha-1} - 1}$

$\Rightarrow \{S_n\}$ is bounded and also monotonic. Hence we conclude that $\sum \frac{1}{n^\alpha}$ is convergent when $\alpha > 1$.

If $\alpha \le 1$ then

$$n^\alpha \le n \quad \forall \; n \ge 1$$

$$\Rightarrow \frac{1}{n^\alpha} \ge \frac{1}{n} \quad \forall \; n \ge 1$$

$\because \sum \frac{1}{n}$ is divergent therefore $\sum \frac{1}{n^\alpha}$ is divergent when $\alpha \le 1$.

Theorem

Let $a_n > 0$, $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lambda \neq 0$ then the series $\sum a_n$ and $\sum b_n$ behave alike.

Proof

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lambda$

$$\Rightarrow \left| \frac{a_n}{b_n} - \lambda \right| < \varepsilon \quad \forall \ n \geq n_0 .$$

Use $\varepsilon = \frac{\lambda}{2}$

$$\Rightarrow \left| \frac{a_n}{b_n} - \lambda \right| < \frac{\lambda}{2} \quad \forall \ n \geq n_0 .$$

$$\Rightarrow \lambda - \frac{\lambda}{2} < \frac{a_n}{b_n} < \lambda + \frac{\lambda}{2}$$

$$\Rightarrow \frac{\lambda}{2} < \frac{a_n}{b_n} < \frac{3\lambda}{2}$$

then we got

$$a_n < \frac{3\lambda}{2} b_n \quad \text{and} \quad b_n < \frac{2}{\lambda} a_n$$

Hence by comparison test we conclude that $\sum a_n$ and $\sum b_n$ converge or diverge together.

Example

To check $\sum \frac{1}{n} \sin^2 \frac{x}{n}$ diverges or converges consider

$$a_n = \frac{1}{n} \sin^2 \frac{x}{n} \quad \text{and take} \quad b_n = \frac{1}{n^3}$$

then $\frac{a_n}{b_n} = n^2 \sin^2 \frac{x}{n}$

$$= \frac{\sin^2 \frac{x}{n}}{\frac{1}{n^2}} \quad = x^2 \left(\frac{\sin \frac{x}{n}}{\frac{x}{n}} \right)^2$$

Applying limit as $n \to \infty$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} x^2 \left(\frac{\sin \frac{x}{n}}{\frac{x}{n}} \right)^2 = x^2 \left(\lim_{n\to\infty} \frac{\sin \frac{x}{n}}{\frac{x}{n}} \right)^2 = x^2 (1) = x^2$$

$\Rightarrow \sum a_n$ and $\sum b_n$ have the similar behavior $\forall$ finite values of x except $x = 0$.

Since $\sum \frac{1}{n^3}$ is convergent series therefore the given series is also convergent for finite values of x except $x = 0$.

Theorem (Cauchy Condensation Test)

Let $a_n \geq 0$, $a_n > a_{n+1}$ $\forall$ $n \geq 1$, then the series $\sum a_n$ and $\sum 2^{n-1} a_{2^{n-1}}$ converges or diverges together.

Proof

Let us suppose

$$S_n = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots\ldots + a_n$$

and $$t_n = a_1 + 2a_2 + 2^2 a_{2^2} + \ldots\ldots\ldots\ldots\ldots + 2^{n-1} a_{2^{n-1}}.$$

$\because$ $a_n \geq 0$ and $n < 2^{n-1} < 2^n - 1$

$\therefore$ $S_n < S_{2^{n-1}} < S_{2^n - 1}$ for $n > 2$

then

$$S_{2^n - 1} = a_1 + a_2 + a_3 + \ldots\ldots + a_{2^n - 1}$$

$$= a_1 + \left(a_2 + a_3\right) + \left(a_4 + a_5 + a_6 + a_7\right) + \ldots\ldots. + \left(a_{2^{n-1}} + a_{2^{n-1}+1} + a_{2^{n-1}+2} + \ldots.. + a_{2^n - 1}\right)$$

$$< a_1 + \left(a_2 + a_2\right) + \left(a_4 + a_4 + a_4 + a_4\right) + \ldots\ldots. + \left(a_{2^{n-1}} + a_{2^{n-1}} + a_{2^{n-1}} + \ldots.. + a_{2^{n-1}}\right)$$

$$< a_1 + 2a_2 + 2^2 a_4 + \ldots\ldots. + 2^{n-1} a_{2^{n-1}} = t_n$$

$\Rightarrow$ $S_n < t_n$

$\Rightarrow$ $S_n < t_n < 2S_{2^n}$ ………… (i)

Now consider

$$S_{2^n} = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots\ldots + a_{2^n}$$

$$= a_1 + a_2 + \left(a_3 + a_4\right) + \left(a_5 + a_6 + a_7 + a_8\right) + \ldots\ldots. + \left(a_{2^{n-1}+1} + a_{2^{n-1}+2} + a_{2^{n-1}+3} + \ldots. + a_{2^n}\right)$$

$$> \frac{1}{2} a_1 + a_2 + \left(a_4 + a_4\right) + \left(a_8 + a_8 + a_8 + a_8\right) + \ldots\ldots + \left(a_{2^n} + a_{2^n} + a_{2^n} + \ldots.. + a_{2^n}\right)$$

$$= \frac{1}{2} a_1 + a_2 + 2a_4 + 2^2 a_8 + \ldots\ldots\ldots\ldots\ldots + 2^{n-1} a_{2^n}$$

$$= \frac{1}{2}\left(a_1 + 2a_2 + 2^3 a_4 + 2^3 a_8 + \ldots\ldots\ldots\ldots\ldots + 2^n a_{2^n}\right)$$

$\Rightarrow$ $S_{2n} > \frac{1}{2} t_n$ ………… (ii)

$\Rightarrow$ $2S_{2n} > t_n$

From (i) and (ii) we see that the sequence S_n and t_n are either both bounded or both unbounded, implies that $\sum a_n$ and $\sum 2^{n-1} a_{2^{n-1}}$ converges or diverges together.

Example

Consider the series $\sum \frac{1}{n^p}$

If $p \le 0$ then $\lim\limits_{n\to\infty} \frac{1}{n^p} \ne 0$

therefore the series diverges when $p \le 0$.

If $p > 0$ then the condensation test is applicable and we are lead to the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{2^{kp-k}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^{(p-1)k}} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{(p-1)}} \right)^k$$

$$= \sum_{k=0}^{\infty} 2^{(1-p)k}$$

Now $2^{1-p} < 1$ iff $1-p < 0$ i.e. when $p > 1$

And the result follows by comparing this series with the geometric series having common ratio less than one.

The series diverges when $2^{1-p} = 1$ (i.e. when $p = 1$)

The series is also divergent if $0 < p < 1$.

Example

If $p > 1$, $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges and

If $p \le 1$ the series is divergent.

$\because$ $\{\ln n\}$ is increasing $\therefore$ $\left\{ \frac{1}{n \ln n} \right\}$ decreases

and we can use the condensation test to the above series.

We have $a_n = \frac{1}{n(\ln n)^p}$

$$\Rightarrow \quad a_{2^n} = \frac{1}{2^n (\ln 2^n)^p} \qquad \Rightarrow \quad 2^n a_{2^n} = \frac{1}{(n \ln 2)^p}$$

$\Rightarrow$ we have the series

$$\sum 2^n a_{2^n} = \sum \frac{1}{(n \ln 2)^p} = \frac{1}{(\ln 2)^p} \sum \frac{1}{n^p}$$

which converges when $p > 1$ and diverges when $p \le 1$.

Example

Consider $\sum \frac{1}{\ln n}$

Since $\{\ln n\}$ is increasing there $\left\{\frac{1}{\ln n}\right\}$ decreases.

And we can apply the condensation test to check the behavior of the series

$$\because \; a_n = \frac{1}{\ln n} \qquad \therefore \; a_{2^n} = \frac{1}{\ln 2^n}$$

so $\quad 2^n a_{2^n} = \frac{2^n}{\ln 2^n} \quad \Rightarrow \quad 2^n a_{2^n} = \frac{2^n}{n \ln 2}$

since $\quad \frac{2^n}{n} > \frac{1}{n} \qquad \forall \; n \geq 1$

and $\sum \frac{1}{n}$ is diverges therefore the given series is also diverges.

Alternating Series

A series in which the successive terms have opposite signs is known as alternating series.

Theorem (Alternating Series Test or Leibniz Test)

Let $\{a_n\}$ be a decreasing sequence of positive numbers such that $\lim\limits_{n\to\infty} a_n = 0$ then the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots\ldots\ldots\ldots\ldots$ converges.

Proof

Looking at the odd numbered partial sums of this series we find that

$$S_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \ldots\ldots\ldots + (a_{2n-1} - a_{2n}) + a_{2n+1}$$

Since $\{a_n\}$ is decreasing therefore all the terms in the parenthesis are non-negative

$$\Rightarrow S_{2n+1} > 0 \quad \forall \; n$$

Moreover

$$\begin{aligned} S_{2n+3} &= S_{2n+1} - a_{2n+2} + a_{2n+3} \\ &= S_{2n+1} - \left(a_{2n+2} - a_{2n+3}\right) \end{aligned}$$

Since $\quad a_{2n+2} - a_{2n+3} \geq 0 \quad$ therefore $\quad S_{2n+3} \leq S_{2n+1}$

Hence the sequence of odd numbered partial sum is decreasing and is bounded below by zero. (as it has +ive terms)

It is therefore convergent.

Thus S_{2n+1} converges to some limit l (say).

Now consider the even numbered partial sum. We find that

$$S_{2n+2} = S_{2n+1} - a_{2n+2}$$

and

$$\begin{aligned} \lim_{n\to\infty} S_{2n+2} &= \lim_{n\to\infty}\left(S_{2n+1} - a_{2n+2}\right) \\ &= \lim_{n\to\infty} S_{2n+1} - \lim_{n\to\infty} a_{2n+2} \\ &= l - 0 = l \qquad \because \; \lim_{n\to\infty} a_n = 0 \end{aligned}$$

so that the even partial sum is also convergent to l.
$\Rightarrow$ both sequences of odd and even partial sums converge to the same limit.
Hence we conclude that the corresponding series is convergent.

Absolute Convergence

$\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges.

Theorem

An absolutely convergent series is always convergent.

Proof:

If $\sum |a_n|$ is convergent then for a real number $\varepsilon > 0$, $\exists$ a positive integer n_0 such that

$$\left|\sum_{i=m+1}^{n} a_i\right| < \sum_{i=m+1}^{n} |a_i| < \varepsilon \qquad \forall\ n, m > n_0$$

$\Rightarrow$ the series $\sum a_n$ is convergent. (Cauchy Criterion has been used)

Note

The converse of the above theorem does not hold.

e.g. $\sum \frac{(-1)^{n+1}}{n}$ is convergent but $\sum \frac{1}{n}$ is divergent.

Theorem (The Root Test)

Let $\lim_{n\to\infty} Sup |a_n|^{1/n} = p$

Then $\sum a_n$ converges absolutely if $p < 1$ and it diverges if $p > 1$.

Proof

Let $p < 1$ then we can find the positive number $\varepsilon > 0$ such that $p + \varepsilon < 1$

$$\Rightarrow |a_n|^{1/n} < p + \varepsilon < 1 \qquad \forall\ n > n_0$$

$$\Rightarrow |a_n| < (p + \varepsilon)^n < 1$$

$\because \sum (p+\varepsilon)^n$ is convergent because it is a geometric series with $|r| < 1$.

$\therefore \sum |a_n|$ is convergent

$\Rightarrow \sum a_n$ converges absolutely.

Now let $p > 1$ then we can find a number $\varepsilon_1 > 0$ such that $p - \varepsilon_1 > 1$.

$$\Rightarrow |a_n|^{1/n} > p + \varepsilon > 1$$

$\Rightarrow |a_n| > 1$ for infinitely many values of n.

$$\Rightarrow \lim_{n\to\infty} a_n \neq 0$$

$\Rightarrow \sum a_n$ is divergent.

Note:

The above test give no information when $p=1$.

e.g. Consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.

For each of these series $p=1$, but $\sum \frac{1}{n}$ is divergent and $\sum \frac{1}{n^2}$ is convergent.

Theorem (Ratio Test)

The series $\sum a_n$

(i) Converges if $\lim_{n\to\infty} Sup \left|\frac{a_{n+1}}{a_n}\right| < 1$

(ii) Diverges if $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for $n \geq n_0$, where n_0 is some fixed integer.

Proof

If (i) holds we can find $\beta < 1$ and integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta \quad \text{for } n \geq N$$

In particular

$$\left|\frac{a_{N+1}}{a_N}\right| < \beta$$

$$\Rightarrow |a_{N+1}| < \beta |a_N|$$

$$\Rightarrow |a_{N+2}| < \beta |a_{N+1}| < \beta^2 |a_N|$$

$$\Rightarrow |a_{N+3}| < \beta^3 |a_N|$$

....

....

....

$$\Rightarrow |a_{N+p}| < \beta^p |a_N|$$

$$\Rightarrow |a_n| < \beta^{n-N} |a_N| \qquad \text{we put } N+p=n.$$

i.e. $|a_n| < |a_N| \, \beta^{-N} \beta^n$ for $n \geq N$.

$\because \sum \beta^n$ is convergent because it is geometric series with common ration <1.

Therefore $\sum a_n$ is convergent (by comparison test)

Now if

$$|a_{n+1}| \geq |a_n| \qquad \text{for } n \geq n_0$$

then $\lim_{n\to\infty} a_n \neq 0$

$\Rightarrow \sum a_n$ is divergent.

Note

The knowledge $\left|\frac{a_{n+1}}{a_n}\right| = 1$ implies nothing about the convergent or divergent of series.

Example

Consider the series $\sum a_n$ with $a_n = \left[\frac{n}{n+1} - \left(\frac{n}{n+1}\right)^{n+1}\right]^{-n}$

$$\because \frac{n}{n+1} < 1 \qquad \therefore \quad a_n > 0 \quad \forall \ n.$$

Also $(a_n)^{\frac{1}{n}} = \left[\frac{n}{n+1} - \left(\frac{n}{n+1}\right)^{n+1}\right]^{-1}$

$$= \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n}{n+1}\right)^{n}\right]^{-1} = \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{n+1}{n}\right)^{-n}\right]^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \lim_{n\to\infty}\left[1 - \left(1+\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= 1\cdot\left[1-e^{-1}\right]^{-1} = \left[1-\frac{1}{e}\right]^{-1} = \left[\frac{e-1}{e}\right]^{-1} = \left[\frac{e}{e-1}\right] > 1$$

$\Rightarrow$ the series is divergent.

Theorem

Suppose that $\sum a_n$ is convergent and that $\{b_n\}$ is monotonic convergent sequence then $\sum a_n b_n$ is also convergent.

Proof

Suppose $\{b_n\}$ is decreasing and it converges to b.

Put $c_n = b_n - b$

$\Rightarrow$ $c_n \geq 0$ and $\lim\limits_{n\to\infty} c_n = 0$

$\because$ $\sum a_n$ is convergent

$\therefore$ $\{S_n\}$, $S_n = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots + a_n$ is convergent

$\Rightarrow$ It is bounded

$\Rightarrow$ $\sum a_n c_n$ is bounded.

$\because$ $a_n b_n = a_n c_n + a_n b$ and $\sum a_n c_n$ and $\sum a_n b$ are convergent.

$\therefore$ $\sum a_n b_n$ is convergent.

Now if $\{b_n\}$ is increasing and converges to b then we shall put $c_n = b - b_n$.

Example

$\sum \frac{1}{(n \ln n)^{\alpha}}$ is convergent if $\alpha > 1$ and divergent if $\alpha \le 1$.

To see this we proceed as follows

$$a_n = \frac{1}{(n \ln n)^{\alpha}}$$

Take $b_n = 2^n a_{2^n} = \frac{2^n}{\left(2^n \ln 2^n\right)^{\alpha}} = \frac{2^n}{\left(2^n\ n \ln 2\right)^{\alpha}}$

$$= \frac{2^n}{2^{n\alpha}\, n^{\alpha} \left(\ln 2\right)^{\alpha}} = \frac{1}{2^{n\alpha - n}\, n^{\alpha} \left(\ln 2\right)^{\alpha}}$$

$$= \frac{1}{\left(\ln 2\right)^{\alpha}} \cdot \frac{\left(\frac{1}{2}\right)^{(\alpha-1)n}}{n^{\alpha}}$$

Since $\sum \frac{1}{n^{\alpha}}$ is convergent when $\alpha > 1$ and $\left(\frac{1}{2}\right)^{(\alpha-1)n}$ is decreasing for $\alpha > 1$ and it converges to 0. Therefore $\sum b_n$ is convergent

$\Rightarrow \sum a_n$ is also convergent.

Now $\sum b_n$ is divergent for $\alpha \le 1$ therefore $\sum a_n$ diverges for $\alpha \le 1$.

Example

To check $\sum \frac{1}{n^{\alpha} \ln n}$ is convergent or divergent.

We have $a_n = \frac{1}{n^{\alpha} \ln n}$

Take $b_n = 2^n a_{2^n} = \frac{2^n}{(2^n)^{\alpha} (\ln 2^n)} = \frac{2^n}{2^{n\alpha} (n \ln 2)}$

$$= \frac{1}{\ln 2} \cdot \frac{2^{(1-\alpha)n}}{n} = \frac{1}{\ln 2} \cdot \frac{\left(\frac{1}{2}\right)^{(\alpha-1)n}}{n}$$

$\because \sum \frac{1}{n}$ is divergent although $\left\{\left(\frac{1}{2}\right)^{n(\alpha-1)}\right\}$ is decreasing, tending to zero for $\alpha > 1$

therefore $\sum b_n$ is divergent.

$\Rightarrow \sum a_n$ is divergent.

The series also divergent if $\alpha \le 1$.

i.e. it is always divergent.